PAT Handout

Mathematics

Contents

July 3, 2020

§0 Introduction

The **Physics Aptitude Test (PAT)** is a two hour long examination in which approximately half of the problems presented aim to assess your mathematical aptitude. The fundamental knowledge required to solve these problems does not go deeper than the mathematics one may have already come across in studying A-level mathematics and Physics by the time the PAT takes place in late October. That is not, however, to say they are easy. Often the problems require small insights and tricks to test your mastery of the subject. This handout hopes to draw back the curtain on these problems by showing you certain tricks, as well as providing a re-cap of the content one is expected to know for the exam.

Below is an index of the syllabus as listed on the University of Oxford's Department of Physics website:

Syllabus for the Mathematics content of Physics Aptitude Test

Elementary mathematics:

- Knowledge of elementary mathematics, in particular topics in arithmetic, geometry including coordinate geometry, and probability, will be assumed. Questions may require the manipulation of mathematical expressions in a physical context.

Algebra:

- Knowledge of the properties of polynomials, including the solution of quadratics either using a formula or by factorising.
- Graph sketching including the use of differentiation to find stationary points.
- Transformations of variables.
- Solutions to inequalities.
- Elementary trigonometry including relationships between sine, cosine and tangent (sum and difference formulae will be stated if required).
- Properties of logarithms and exponentials and how to combine logarithms, e.g. $\log(a)+\log(b) = \log(ab)$.
- Knowledge of the formulae for the sum of arithmetic and geometric progressions to n (or infinite) terms.
- Use of the binomial expansion for expressions such as $(a + bx)^n$, using only positive integer values of n.

Calculus:

- Differentiation and integration of polynomials including fractional and negative powers.
- Differentiation to find the slope of a curve, and the location of maxima and minima.
- Integration as the reverse of differentiation and as finding the area under a curve.
- Simplifying integrals by symmetry arguments including use of the properties of even and odd functions (where an even function has $f(x) = f(-x)$, an odd function has $f(-x) = -f(x)$).

As a final note, the PAT is supposed to be difficult. It's purpose is to differentiate between students in ways standardised testing cannot. For that reason it is vital that you are not discouraged when you find yourself struggling to make progress on the questions. For many it may be your first time being exposed to such questions, but with practice and determination you will become more adapt and efficient at coming up with solutions and progressing through the paper.

§1 Geometry

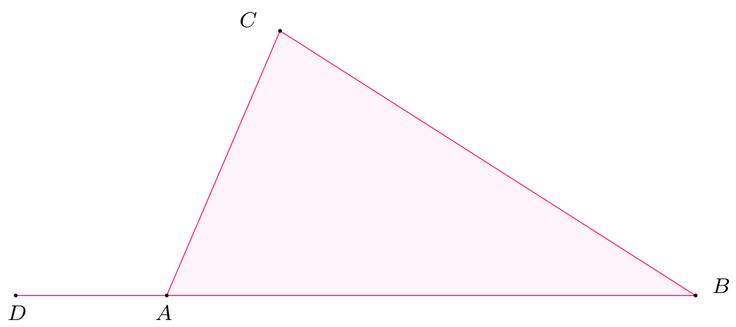
In the PAT you may potentially come across two areas of geometry; *analytic* geometry, that is, geometry involving coordinates, trigonometry and so forth, and *euclidean* geometry - geometry based on axioms[1], theorems and so on. This section will cover some of the things you might want to keep in mind while approaching such problems. All of the euclidean geometry needed you'll have already covered in doing GCSE mathematics, but we'll refresh your memory anyway!

§1.1 Euclidean Geometry

To begin with, lets consider some basic laws about triangles.

Theorem 1.1 (Exterior angle of a triangle)
The external angle $\angle DAC$ is equal to the sum of the two opposite internal angles of the triangle, $\angle ABC$ and $\angle BCA$:

$$\angle DAC = \angle ABC + \angle BCA$$



Proof. Since we know that the angles inside a triangle sum up to 180° we can form the following equation:

$$\angle CAB + \angle ABC + \angle BCA = 180^\circ \tag{1}$$

We also observe that $\angle DAC$ and $\angle CAB$ are supplementary angles. This means that they both add up to 180°. So we can form a second equation:

$$\angle DAC + \angle CAB = 180^\circ \tag{2}$$

As 1 and 2 share a common term in $\angle CAB$, we see that we can substitute $\angle CAB = 180 - \angle DAC$ into our first equation. This gives:

$$(180^\circ - \angle DAC) + \angle ABC + \angle BCA = 180^\circ \tag{3}$$

Finally, simplifying results in what we set out to prove:

$$\angle ABC + \angle BCA = \angle DAC$$

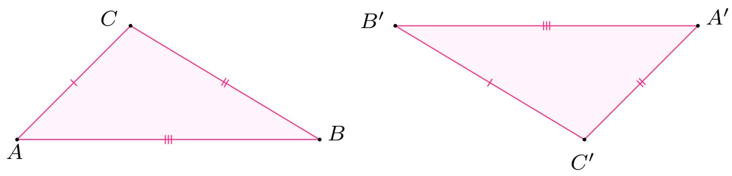
□

[1]These are fundamental claims that we take to be true as they cannot be proven under any known - or compatible - logical formalisation of mathematics. See this Wikipedia article for further reading.

We will now introduce the idea of *congruence*. We say two triangles are congruent if they are the same. This may sound a little cryptic. What I mean is that given two congruent triangles, the lengths of the sides of the first triangle would be the same as the corresponding sides to the second triangle. Similarly, the measure of the corresponding are also equal. A significant consequence of this is that congruence is preserved under linear transformations such as reflections and rotations.

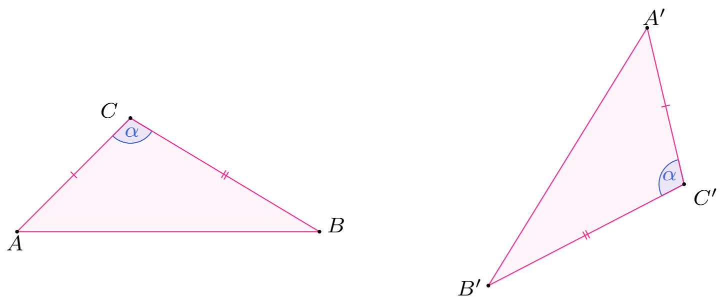
Here we'll show you how to figure out if triangles are congruent.

Definition 1.2 (Side-Side-Side Congruence - Axiom)**.** Should two triangles have correspondingly equal lengths, then we say the triangles are congruent.



> **Remark 1.3**
> Naturally, given two triangles that are congruent,the the corresponding angles of those congruent triangles must similarly be equal.

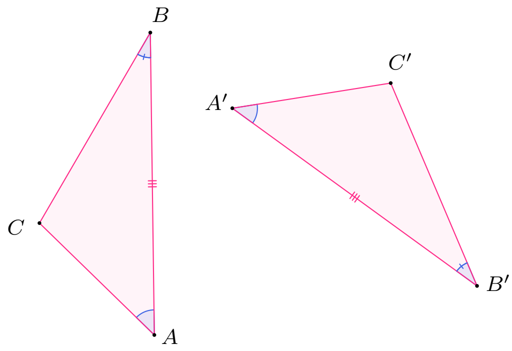
Definition 1.4 (Side-Angle-Side Congruence - Axiom)**.** If two triangles have *two* correspondingly equal sides, and the angle between them are also corresponding equal, then we say the triangles are congruent.



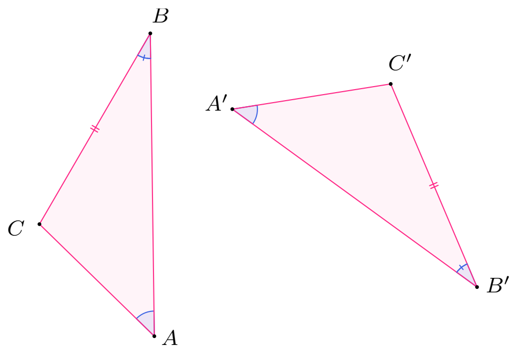
> **Remark 1.5**
> Since $AC = A'C'$, $BC = B'C'$ and $\angle ACB = \angle A'C'B'$, we have established that $\triangle ABC$ is congruent to $\triangle A'B'C'$. Often we'll use the notation $\triangle ABC \cong \triangle A'B'C'$ to denote this fact.

> **Corollary 1.6**
> As $\triangle ABC \cong \triangle A'B'C'$, it must also be the case that $\angle CAB = \angle C'A'B'$ and $\angle ABC = \angle A'B'C'$.

Definition 1.7 (Angle-Side-Angle Congruence - Axiom)**.** Should two angles of a triangle, and the side between them be equal to the corresponding angles and side of another triangle, then we say the two triangles are congruent.



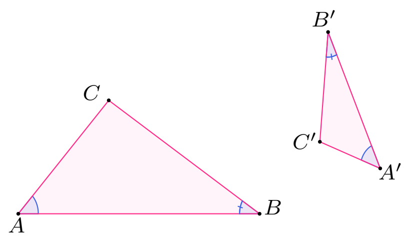
Definition 1.8 (Angle-Angle-Side Congruence - Axiom)**.** If two triangles have two corresponding angles of equal measure, and another corresponding side of equal length, which is not between the aforementioned angles, then we the triangles are congruent.



Question 1.9. Side-Side-Angle relationships between triangles do not result in congruence. Using a compass and ruler, try justifying this by constructing two triangles that share a Side-Side-Angle relationship but are *not* congruent.

As we've seen, under operations such as reflections and rotations, the idea of distance is preserved, this means that the sides of the triangles, and the measure of the angles, will remain equal under such transformations. This is not the case for dilations, however. These are transformations that scale or change the size of the triangle. Under dilations, the lengths of the sides of the triangles are not preserved - however, angles and also the *ratios* of the sides of the triangles are. Under these circumstances we introduce a new idea - similarity.

Theorem 1.10 (Angle-Angle Similarity)
If two triangles have equal corresponding angles then we say the two triangles are similar.



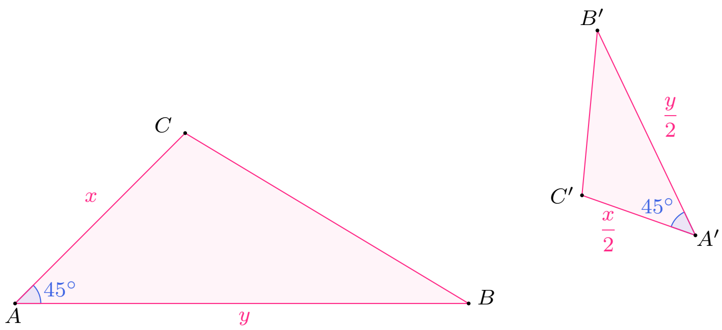
Corollary 1.11
The ratio of the sides still still preserved under dilation. As a consequence we have the following result:

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

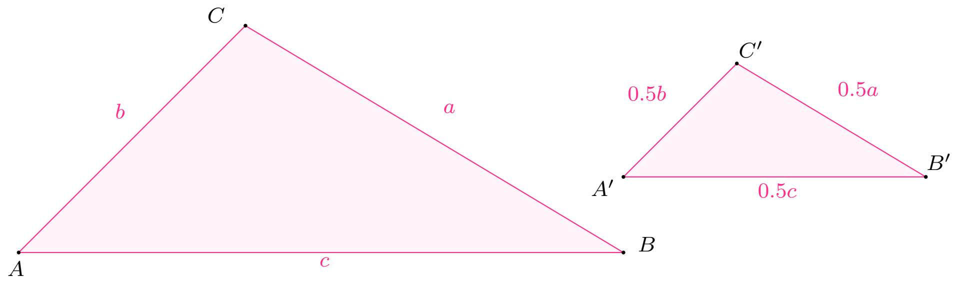
Proof. **TO DO** □

Remark 1.12
Often we'll use the notation $\triangle ABC \sim \triangle A'B'C'$ to denote similarity.

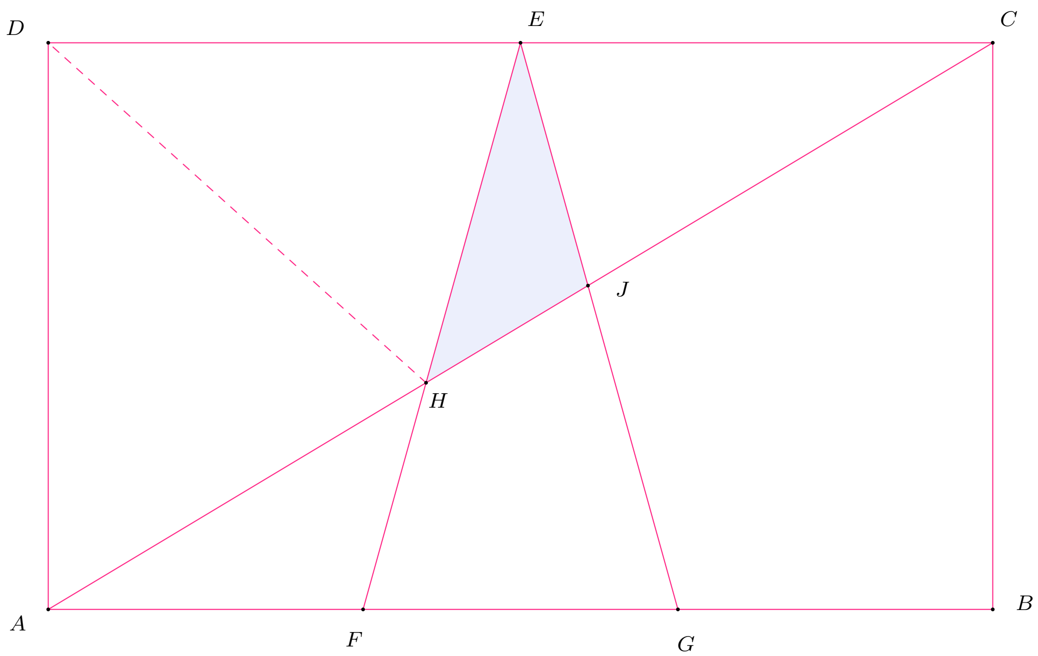
Theorem 1.13 (Side-Angle-Side Similarity)
Supposing the ratio of two sides in one triangle is equal to the ratio of the ratio of the corresponding sides in another triangle, and the measure of the angle between the two sides in ratio is equal to it's counter in the other triangle, then we say that the triangle are similar.



Theorem 1.14 (Side-Side-Side Similarity)
If each side of a triangle is a scalar multiple of the corresponding sides of another triangle, then we can say the triangles are similar.

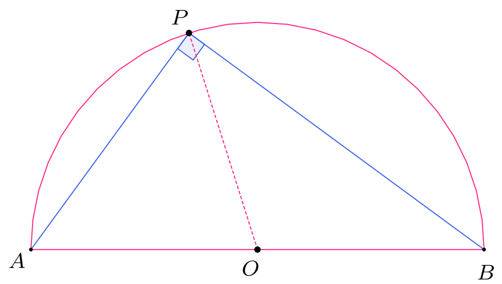


Question 1.15 (2001 AMC 12 #22)**.** In rectangle $ABCD$, points F and G lie on AB so that $AF = FG = GB$ and E is the midpoint of DC. Also AC intersects EF at H and EG at J. The area of the rectangle $ABCD$ is 70. Find the area of triangle EHJ.

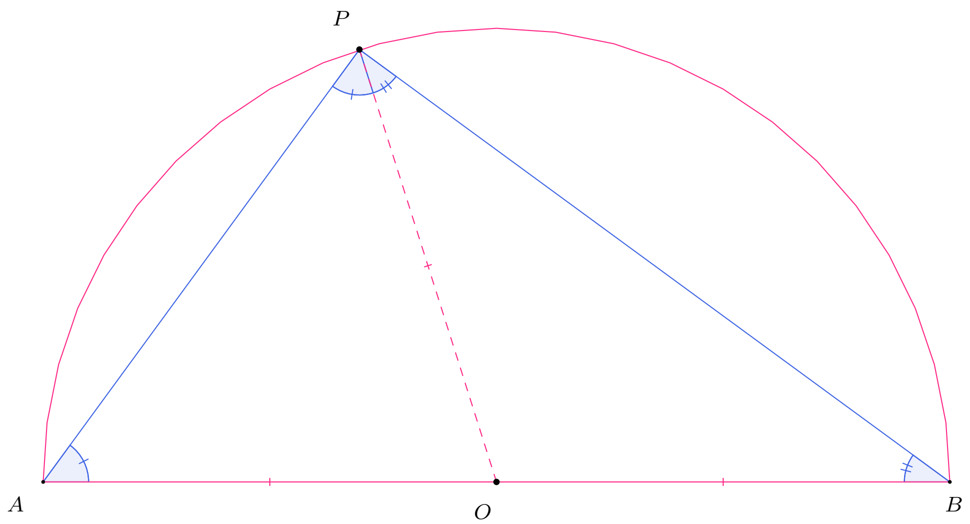


Now that we have discussed triangles in some detail, we can then go onto discuss theorems involving circles (which often play nicely together with triangular theorems). A lot of this will be a refresher from GCSE, but again it may well pop up in new and unexpected ways.

Theorem 1.16 (Angle in a Semi-circle)
For a circle with diameter AB, and a point P on the circumference of the circle, we have $\angle APB = 90°$



Proof. As $OA = OP = OB$, we have two isosceles triangles in $\triangle AOP$ and $\triangle OBP$. Hence $\angle PAO = \angle OPA$ and $\angle OBP = \angle BPO$, which we will name x and y respectively. We wish to show that $\angle OPA + \angle BPO = 90°$- which is the same as showing $x + y = 90°$

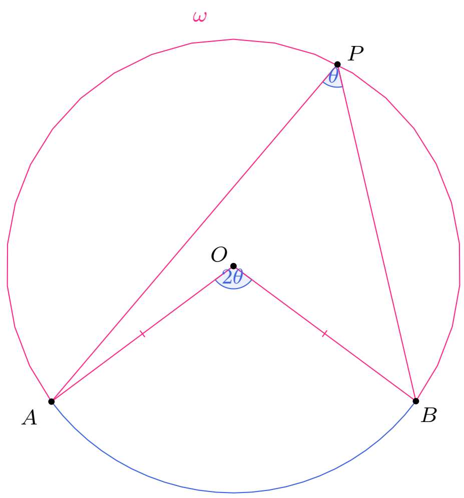


Observe that the angles the angles in a triangle must add up to 180°. Therefore we have the following:

$$\begin{aligned} \angle OAP + \angle APO + \angle OBP + \angle BPO &= 180° \\ 2x + 2y &= 180° \\ x + y &= 90° \\ \therefore \angle OPA + \angle BPO &= 90° \end{aligned} \tag{4}$$

As required. □

Theorem 1.17 (Inscribed Angle Theorem)
Given an arc AB on circle ω, and another point P inscribed on ω, we have $\angle APB = 2\angle AOB$

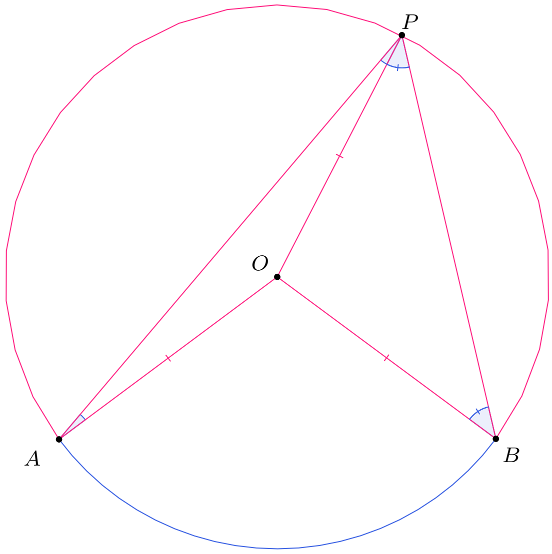


Proof. We want to show that $\angle APB = 2\angle AOB$. By constructing segment OP we see that we have two isosceles triangles in $\triangle OPA$ and $\triangle OBP$. Hence we have $\angle OPA = \angle PAO$ and similarly, $\angle BPO = \angle OBP$. Let these respective angles equal x and y respectively, then we wish to show that $\angle AOB = 2(x + y)$

Notice that $\angle AOP = 180° - 2x$ and $\angle POB = 180° - 2y$ and since $\angle AOB + \angle AOP + \angle POB = 360°$ we have the following:

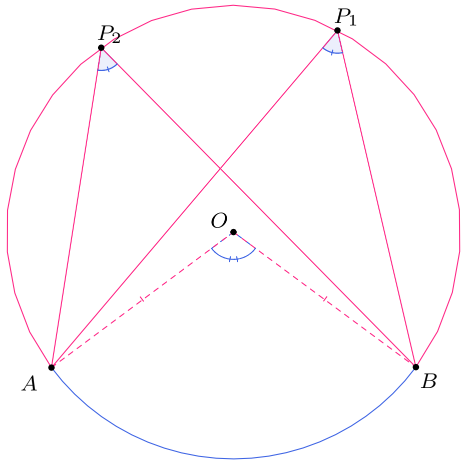
$$\begin{aligned} \angle AOB + (180° - 2x) + (180° - 2y) = 360° \\ \therefore \angle AOB = 2(x + y) \end{aligned} \tag{5}$$

And we're done.



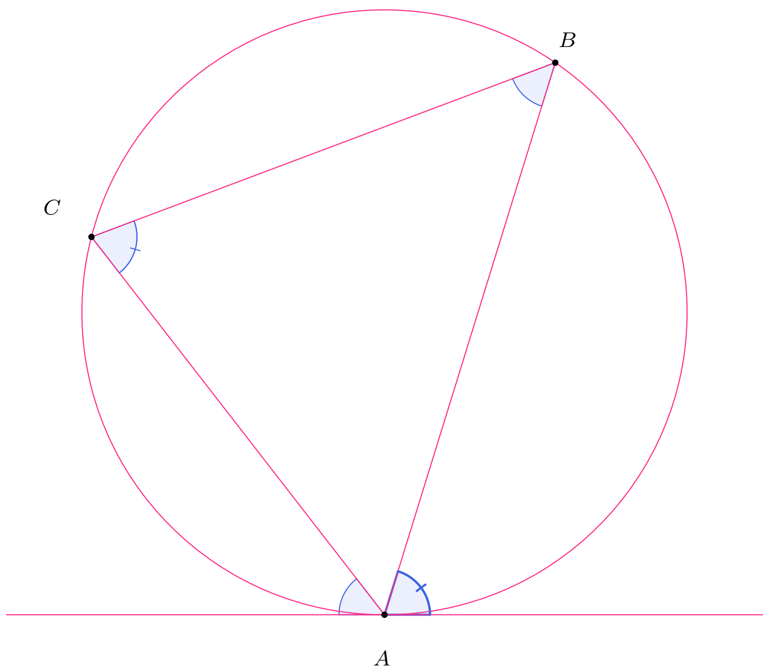
□

Theorem 1.18 (Angles inscribed to the same segment)
Given two distinct point on a circle inscribed by arc AB, then $\angle AP_1B = \angle AP_2B$



Proof. This is an immediate consequence of 1.17 □

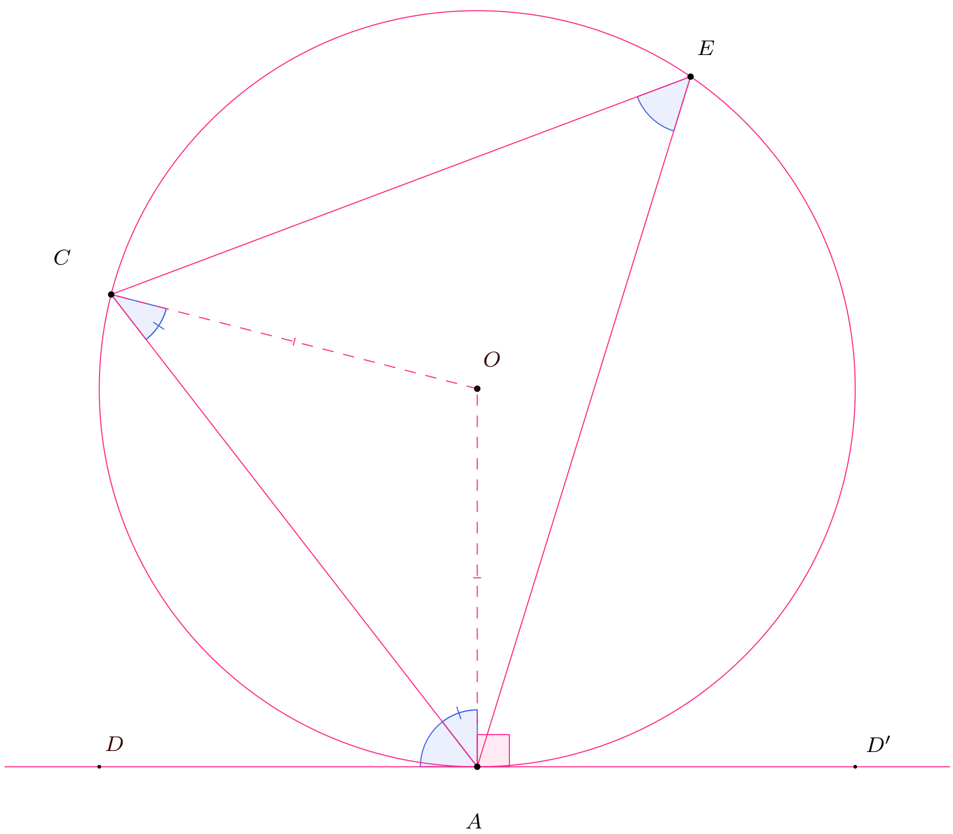
Theorem 1.19 (Alternate Segment Theorem)
For an angle formed by a line tangent to a circle and a chord from the point of tangency, the measure of that angle is equal to the angle in the alternate segment.



Proof. Since $\triangle AOC$ is isosceles, we have $\angle AOC = \angle ACO$, and thus $\angle AOC = 180° - (\angle AC0 + \angle OAC$. As $\angle DAO$ is a right angle we also have $\angle CAO = \angle OCA = 90° - \angle DAC$. This means that we have:

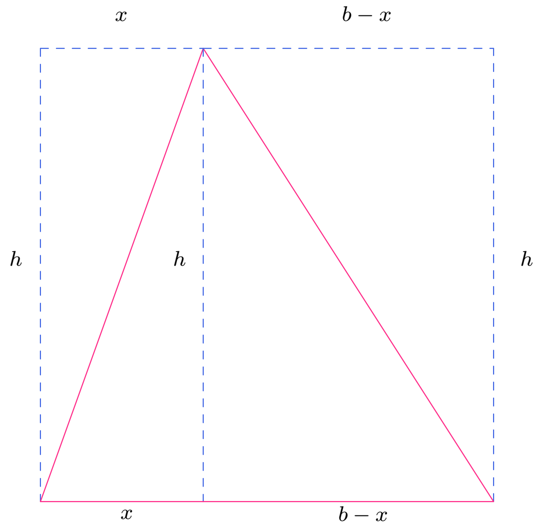
$$\begin{aligned} \angle AOC &= 180 - 2(90 - \angle DAO) \\ &= 2\angle DAO \end{aligned} \tag{6}$$

Therefore by 1.17 we have $\angle DAC = \angle ABC$, as required.



□

We all know how to work out the area of the triangle - sometimes we use $\frac{1}{2}(A)(B)\sin(C)$ or $\frac{1}{2}bh$. Here we'll show two other ways to work out the area of the triangle. The first is very easy to visualise. Instead of working out the area of the triangle directly, we draw a square around it, then work out the area of the smaller triangles:



Remark 1.20
We use $[ABC]$ to denote the area of the polygon - or in this case, triangle - ABC

Hence, $[ABC] = bh - \frac{1}{2}(x)(h) - \frac{1}{2}(h)(b-x)$, and this simplifies to $[ABC] = \frac{1}{2}bh$, as expected. Naturally this is a very trivial and obvious example, but is useful in highlighting the idea. The next idea we'll introduce is a little more advanced and it's a nice little trick to keep in your back pocket.

Theorem 1.21 (Heron's Formula)
For a triangle with sides of a, b, and c, we define

$$s = \frac{a+b+c}{2}$$

then the area of the triangle is

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$$

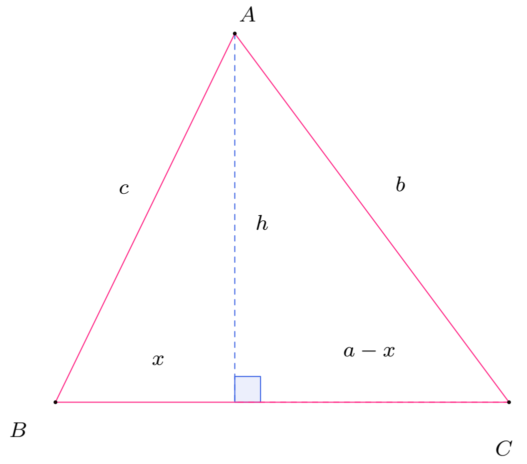
Proof. The proof in essence is just repeated applications of the Pythagorean theorem along with some algebraic manipulation. First consider the diagram.
By Pythagoras, we see that

$$\begin{aligned}(a-x)^2 + h^2 &= c^2 \\ x^2 + h^2 &= b^2\end{aligned} \tag{7}$$

We can now combine the two equations into the form $x^2 - (a-x)^2 = c^2 - b^2$. Therefore,

$$x = \frac{a^2 + c^2 - b^2}{2a}$$

We now want to eliminate h, so consider $h^2 = c^2 - x^2 = (c+x)(c-x)$. We can now substitute our value of x in to get:



$$
\begin{aligned}
h^2 &= \left(c + \frac{a^2+c^2-b^2}{2a}\right)\left(c - \frac{a^2+c^2-b^2}{2a}2a\right) \\
&= \frac{1}{4a^2}\left(2ac + a^2 + c^2 - b^2\right)\left(2ac - a^2 - c^2 + b^2\right) \\
&= \frac{1}{4a^2}\left(b^2 - (a-c)^2\right)\left((a-c)^2 - b^2\right) \\
&= \frac{1}{4a^2}\left[(b-a+c)(b+a-c)\right]\left[(a+c-b)(a+b+c)\right]
\end{aligned} \tag{8}
$$

Now we use the substitution $s = \frac{a+b+c}{2}$ to get:

$$
\begin{aligned}
h^2 &= \frac{1}{4a^2}\left(2s-2a\right)\left(2s-2c\right)\left(2s-2b\right)\left(2s\right)\left(2s\right) \\
&= \frac{4s}{a^2}(s-a)(s-b)(s-c) \\
\therefore h &= \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}
\end{aligned} \tag{9}
$$

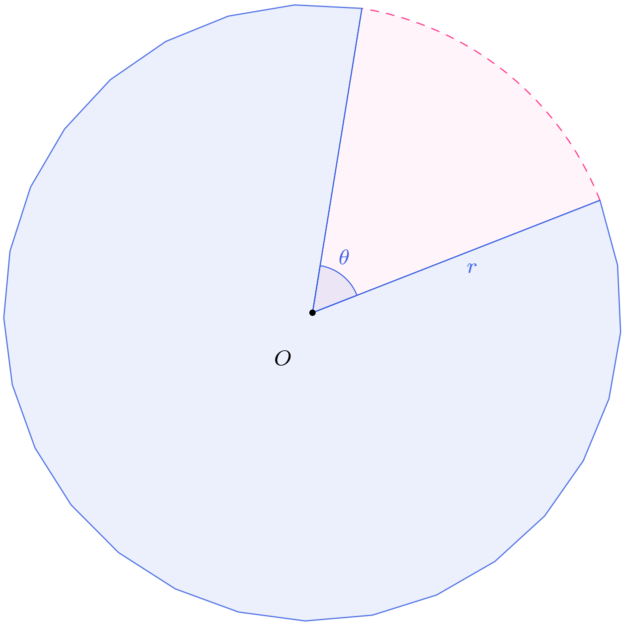
Now, we know the area of $\triangle ABC$ is $\frac{1}{2}(a)(h)$, so substituting our value for h, we get:

$$
\begin{aligned}
[ABC] &= \frac{a}{2} \cdot \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)} \\
&= \sqrt{s(s-a)(s-b)(s-c)}
\end{aligned} \tag{10}
$$

As required. □

As a final addendum to area, here's some small factors about the area of a circle and the area of sectors that may be of use.

Theorem 1.22 (Area of a sector)
The area of a sector is $\frac{1}{2}r^2\theta$

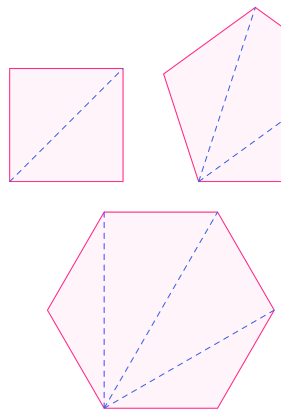


Proof. A proof of this is largely out of the scope of this handout, however, a less rigorous approach to showing that this is true is by first noting that the area of a circle is πr^2, and we only want to work out a *proportion* of said circle. This proportion is equivalent to the angle of the segment over 2π, that is to say we want to work out $\frac{\theta}{2\pi}^{ths}$ of the area of the circle. Thus we have $A = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2}r^2\theta$. □

Theorem 1.23 (Sum of Interior Angles in a Polygon)
The sum of the interior angles of an $n-$sided polygon is $180(n-2)$.

Proof. To prove this we pick a vertex on the polygon and then draw lines to every other vertex on the polygon. The result is that the polygon will now be split up into many triangles. As we know the sum of the angles within a triangle equals $180°$, it suffices to count how many triangles have been made, then multiply that by $180°$.

Since we construct lines to each vertex - but not the two adjacent (because then no triangle is formed), it is clear that there must be two less triangles than there are vertices. So for an $n-$sided polygon, there must be $n-2$ triangles and therefore the sum of said triangles will be $180(n-2)°$, and we're done. □



Corollary 1.24

We can use this information to calculate the measure of each individual internal angle. As there are n such angles, and they sum to $180(n-2)^\circ$, the measure of an internal angle must be $\frac{180(n-2)}{n}^\circ$

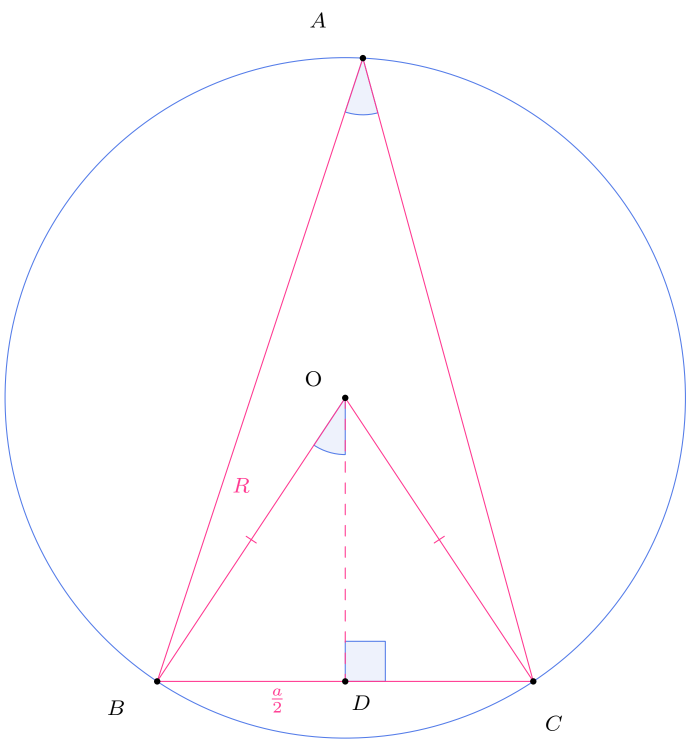
Theorem 1.25 (Law of sines)

For a triangle $\triangle ABC$ we have the following:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \tag{11}$$

Where R is the radius of a circle circumscribing $\triangle ABC$

Proof. The first part of the proof is fairly trivial. We use the fact that $[ABC] = \frac{1}{2}ab\sin C$. Therefore we have $\frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$. Dividing by $\frac{1}{2}abc$ gives the result. For the second part, consider the diagram:



First note that $\triangle ODB \cong ODC$ so $BD = CD = \frac{a}{2}$ and $\angle BOD = \angle COD$. Therefore we have $2\angle BAC = \angle BOC \Rightarrow \angle BOD = \angle COD = \angle A$.

Using trig. on $\triangle BOD$ we get $\sin A = \frac{\frac{a}{2}}{R}$. Therefore, we get $\frac{a}{\sin A} = 2R$ □

Theorem 1.26 (Law of Cosines)
For a triangle $\triangle ABC$ the following holds:

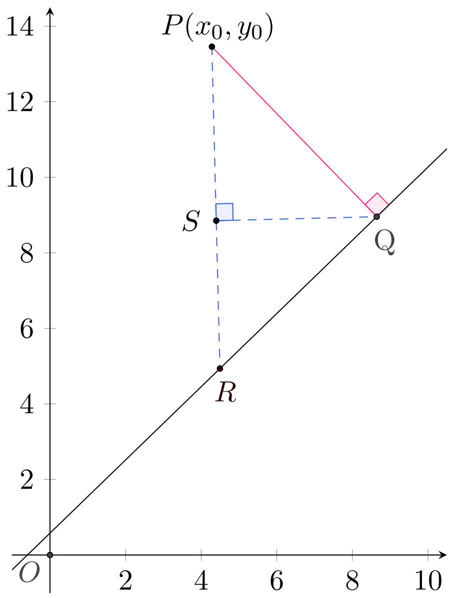
$$c^2 = a^2 + b^2 - 2ab\cos c \tag{12}$$

Exercise 1.27. Try to prove it yourself through applications of Pythagoras! Try considering the case for when we're dealing with a right angle triangle first, then move onto an obtuse or acute triangle.

§1.2 Analytic Geometry

When it comes to coordinates, there isn't a whole lot that you need to know outside $y = mx + c$ is the general form of a straight line, for two points on said line, say (x_0, y_0) and (x_1, y_1), we have $m = \frac{y_0 - y_1}{x_0 - x_1}$. And that the distance between two points is $d = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}$, similarly, the midpoint of two point in the Cartesian plane is $\left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right)$, and finally that the equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$, for a circle centre (h, k) with radius r. We will however share a few neat little tricks that might help you as well as some graph sketching tips.

Theorem 1.28 (Distance between a point and a line)
The distance from a point $P(x_0, y_0)$ and a line of the form $Ax + By + C = 0$ is $d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$



Exercise 1.29. Try to prove it - use the diagram and Pythagoras!

Mastering graph sketching and developing a solid intuition of the behavior of functions will be very helpful in answering questions. When graph sketching it's important to consider a number of things - What happens as $x \to \infty$ and $x \to -\infty$? Are there any asymptotes caused by values in the domain of the function which may result in division by zero? Where are the turning points? What happens when x is very small? And finally, are there any intercepts with the axis'?

Example 1.30

What is the graph of $y = \frac{x^2}{x-3}$

Solution. First we note that as $x \to \infty,\ y \to \infty$ and $x \to -\infty, y \to -\infty$. Secondly we note that at $x = 0,\ y = 0$. We also see that there is an asymptote for $x = 3$ and $y = x + 3$.

Now, finding the turning points:

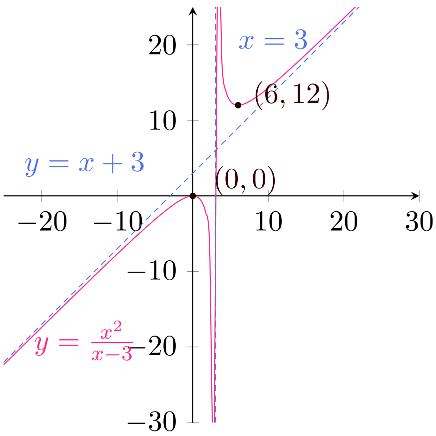
$$y' = \frac{x(x-6)}{(x-3)^2} \tag{13}$$

When $y'0$, we get $x = 0$ and $x = 6$. Thus, we have turning points at $(0, 0)$ and $(6, 12)$. Now we want to find out what type of turning point they are:

$$y'' = \frac{18}{(x-3)^3} \tag{14}$$

For the point $(0, 0)$ we see that it $y'' < 0$ so it is a min point and for $(6, 12)$, $y'' > 0$ and is thus a max point.

We now have a pretty good idea of what this graph looks like:



□

When dealing with quadratic polynomials in particular, you can also use the *discriminant* to gain more insight into what the graph may look like.

Definition 1.31 (Discriminant)**.** We take the *discriminant* of a function to be $b^2 - 4ac$

For a polynomial $ax^2 + bx + c = 0$, if the discriminant is greater than zero, then it has two real roots. If it equals zero, then the quadratic must only touch the x-axis once. This means that the minimum or maximum point of a quadratic must like on said axis. Similarly, if the discriminant is less than zero, no real roots exist. The implication being that the graph is either wholly above the x-axis or below, depending on the value of the leading coefficient a.

As a follow on, a useful method to determine the nature of quadratics is to complete the square.

Theorem 1.32

Suppose we have a quadratic

$$ax^2 + bx + c$$

we can then rewrite this in the following way:

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) \end{aligned}$$

From here, we can see that the minimum (or maximum) occurs at $x = -\frac{b}{2a}$, and when $y = a\left(\frac{c}{a} - \left(\frac{b}{2a}\right)^2\right)$. You can tell whether it is a maxima or minima based on the sign of a:

- if $a > 0$, there is a minima,
- if $a < 0$, there is a maxima

§2 Probability

In the PAT, the probability questions tend to be earlier on in the paper and aren't usually the ones you need to worry about since often the knowledge they require is GCSE level, albeit applied in trickier ways. Here are some key facts you should be confident with:

- For events where outcomes are *equally likely*, $P(\text{outcome}) = \frac{\text{Number of valid outcomes}}{\text{Number of possible outcomes}}$

Example 2.1

Suppose there are a valid outcomes for event A out of a possible N. Then

$$P(A) = \frac{a}{N}$$

Theorem 2.2 (Addition Rule)

If an outcome can occur from two different events A and B, the probability of said outcome, denoted $P(A \cup B)$ is equal to $P(A) + P(B) - P(A \cap B)$, where $P(A \cap B)$ denotes the probability of *both*)

Theorem 2.3 (Bayes Theorem)

For two events, A & B, the probability of A *given* B is denoted by $P(A|B)$. This can be represented in the following ways:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

The way these have been presented isn't always how you'll see the questions, and part of the difficulty is in working out what corresponds to A or B in the above. (Of course, there may also be more events to consider!). Here's a nice example of using Bayes Theorem to give a somewhat counter-intuitive result:

Example 2.4

Suppose there is a quality control machine at a factory where the owners know 1000 faulty units are produced per day. The machine detects faults with 99% accuracy, and 1% of the time you have false positives. If the machine checks 1 million units, what is the probability that any flagged unit is actually faulty?

- Of the 1,000 faulty units, 99% of them are identified - that's 990 units.
- But there are $(1,000,000 - 1,000) \times 0.01$ false positives - 9,990 units.
- Similar to Example 2.1, we have:

$$P(\text{actually faulty}) = \frac{N_{realfaults}}{N_{allfaults}} = \frac{990}{990 + 9990} \approx 9\%$$

- We have essentially used Bayes Theorem here - $P(A \cap B)$ is simply the probability of all *real* faults, and $P(B)$ is the probability of all faults

Probably the biggest takeaway from the probability section however should be how to work out expectation values, and related results.

Theorem 2.5 (Expectation Value)
The expected value of a distribution is defined as:

$$E(x) = \sum xP(X = x)$$
$$= \int xP(X = x)\,dx = \int xf(x)\,dx$$

for discrete and continuous distributions respectively, where $f(x)$ is the probability density function. From here we can define the variance, which is:

$$Var(x) = E(x^2) - (E(x))^2$$

where $E(x^2)$ is:

$$= \sum x^2 P(X = x)$$
$$= \int x^2 P(X = x)\,dx = \int x^2 f(x)\,dx$$

Lastly, we define standard deviation (σ) as

$$\sigma^2 = Var(x)$$

And here we can see this put into action with a set of discrete results

Example 2.6
Consider table 1. From here, we can work out both the expected value and variance.

$$E(x) = 1 \times 0.1 + 2 \times 0.4 + 3 \times 0.45 + 4 \times 0.05 = 2.45$$
$$Var(x) = 1^2 \times 0.1 + 2^2 \times 0.4 + 3^2 \times 0.45 + 4^2 \times 0.05 - 2.45^2 = 0.5475$$

x	1	2	3	4
P(X=x)	0.1	0.4	0.45	0.05

Table 1: random discrete values

Question 2.7. 2005 AMC 12 #11
A letter contains tokens, each representing a numerical amount:

- 2 ones
- 2 fives
- 2 tens
- 2 twenties

Two tokens are drawn at random, without replacement. What is the probability that the sum of the values is greater than 20?

§3 Algebra

When using algebra to solve a problem, more often than not it will require some factoring. Factoring can sometimes be as easy as $x^2 - 2x + 1 = (x-1)^2$, but sometimes, they can be more challenging to spot such as $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$[2].

You might already know the following

$$\begin{aligned} x^2 - y^2 &= (x+y)(x-y) \\ x^3 - y^3 &= (x-y)(x^2 + 2xy + y^2) \\ x^3 + y^3 &= (x+y)(x^2 - 2xy + y^2) \\ x^4 - y^4 &= (x^2 + y^2)(x^2 - y^2) \end{aligned}$$

But what is the factorisation, generally speaking, of $x^n - y^n$ for all values of n?

Theorem 3.1
When n is even:

$$x^{2m} - y^{2m} = (x^m - y^m)(x^m + y^m) \tag{15}$$

When n is odd:

$$x^{2m+1} - y^{2m+1} = (x-y)\sum_{i=0}^{m} x^{n-i}y^i \tag{16}$$

$$x^{2m+1} + y^{2m+1} = (x+y)\sum_{i=0}^{m} x^{n-i}(-y)^i \tag{17}$$

Factoring can also be useful to prove inequalities. The most common principle you'll come across is the fact that $x^2 \geq 0$.

Theorem 3.2 (AM-GM of Two Variables)
$\frac{x+y}{2} \geq \sqrt{xy}$

Proof.

$$\begin{aligned} \frac{x+y}{2} &\geq \sqrt{xy} \\ \left(\frac{x+y}{2}\right)^2 &\geq xy \\ (x+y)^2 &\geq 4xy \\ x^2 - 2xy + y^2 &\geq 0 \\ (x-y)^2 &\geq 0 \end{aligned} \tag{18}$$

As required. □

Remark 3.3
AM-GM stands for Arithmetic Mean - Geometric Mean. Here we are only using it in the context of 2 variables, but it can be generalised to n variables, though this is far outside the scope of the PAT.

[2]This is known as the *Sophie Germain Identity*

When looking to factorise, we can also 'pull out' a factor if we already know it. For instance, if we have $x^3 - 6x^2 + 5x = 0$, and we already know that $x = 0$ is a solution, we can pull out the factor $(x - 0)$. More formally this idea is known as the Factor Theorem.

Theorem 3.4 (Factor Theorem)
For a polynomial $P(x)$, if $x - a$ is a factor of $P(x)$ if, and only if, $P(a) = 0$

To take out a factor, we divide it - this is something you will have covered already in the cause of doing A-level Maths. There is know great way to spot possible factors which can be taken out. It doesn't hurt to do a few test cases to make sure the usual suspects for x are covered. for instance try $P(x)$ for $x = \{-3, -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2, 3\}$. If that doesn't yield anything, your next go to might be *finding* the solutions. In the PAT you'll usually be dealing with quadratics, for any polynomial of a degree greater than 2 more likely than not it's the case that you're expected to *spot* a solution.
Remember, when dealing with polynomials, if it has an even degree, then it might not have any solutions, and that is something you're going to need to consider. A polynomial with an odd degree however is guaranteed to have *at least* one *real* root. For reasons we won't go into (said reasons involve *complex numbers!*). For a cubic equation, if you know that one solution is *not real*, then there must also be another solution which is not real. That is to say that non-real solutions come in pairs. In the case of a cubic equation, if it's the case that one solution is non-real, you therefore know that there is only one real solution.

From experience, it's best not to use the quadratic formulae to find solutions. As we've seen from section 1.2, going through the process of *completing the square* can provide useful insights into the problem, which - typically - the quadratic formula cannot.

Theorem 3.5 (Vieta's Formulas for a polynomial of degree 2)
Given $x^2 + ax + b = (x - p)(x - q)$ (a form we can always put a polynomial in by the fundamental theorem of algebra), we have:

$$\begin{aligned} p + q &= -\frac{b}{a} \\ pq &= \frac{c}{a} \end{aligned} \tag{19}$$

Proof. To prove this simply expand out $(x - p)(x - q)$ and equate like terms with $x^2 + ax + b$ □

Theorem 3.6 (Vieta's Formulas for a polynomial of degree 3)
Given $x^3 + ax^2 + bx + c = (x - p)(x - q)(x - r)$, we have:

$$\begin{aligned} p + q + r &= -\frac{b}{a} \\ pq + pr + qr &= \frac{c}{a} \\ pqr &= -\frac{d}{a} \end{aligned} \tag{20}$$

Proof. Just equate, same as before. □

Another thing that will come up constantly on the PAT is trigonometry. Here's a quick list of formulae you might want to keep in mind. It may look daunting but that's only because we've gone a little overkill -

Pythagorean Identities	**Angle symmetry identities**
$\sin^2\theta + \cos^2\theta = 1$ $\sec^2\theta = 1 + \tan^2\theta$ $\csc^2\theta = 1 + \cot^2\theta$	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$
Angle Addition/Subtraction Identities	$\sin(\pi - \theta) = \sin\theta$
$\sin(\alpha + \beta) + \sin\alpha\cos\beta + \sin\beta\cos\alpha$ $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$	$\sin(\pi - \theta) = \sin(\theta)$ $\cos(\pi - \theta) = -\cos\theta$ $\tan(\pi - \theta) = -\tan\theta$
$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$	**Notable cases**
$\tan(\alpha + \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$ $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$ $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$ $\sin x \pm \sin y = 2\sin\left(\frac{x \pm y}{2}\right)\cos\left(\frac{x \mp y}{2}\right)$	$\sin 2\theta = 2\sin\theta\cos\theta$ $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$ $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
Harmonic Identities	$\cos(2\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$
For $R = \sqrt{A^2 + B^2}$ and $\alpha = \arctan\left(\frac{B}{A}\right)$: $A\sin x + B\cos x = R\sin(x + \alpha)$ $A\cos x + B\sin x = R\cos(x - \alpha)$	$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ $\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1 + \cos\theta} = \frac{1 - \cos\theta}{\sin\theta}$

a lot of these identities can be quickly derived from other ones.

In addition to trigonometry, you should also have a good grasp of using logarithms. Here are the 6 fundamental properties of logarithms:

1. $\log_a b^n = n \log_n b$
2. $\log_a b + \log_a c = \log_a bc$
3. $\log_a b - \log_a c = \log_a \frac{b}{c}$
4. $(\log_a b)(\log_c d) = (\log_a d)(\log_c b)$
5. $\frac{\log_a b}{\log_a c} = \log_c b$
6. $\log_{a^n} b^n = \log_a b$

Example 3.7

Evaluate $P = (\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log 7_8)$

Solution. We can repeatedly use property 4. This yields

$$\begin{aligned} P &= (\log_2 4)(\log_3 3)(\log_4 6)(\log_5 5)(\log_6 8)(\log_7 7) \\ &= (\log_2 4)(\log_4 6)(\log_6 8) \\ &= (\log_2 4)(\log_4 8)(\log_6 6) \\ &= (\log_2 8)(\log_4 4) \\ &= \log_2 8 \\ &= 3 \end{aligned} \tag{21}$$

□

Questions involving the Binomial theorem often pop up on the PAT. The questions might ask you what the coefficient of the x^{3rd} term in the sequence with some extra convolution to make the problem harder. It can also be useful to identify for the purpose of factoring.

Theorem 3.8 (Binomial Theorem)
For a non-negative integer n, we have:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Where we take $\binom{n}{i} = \frac{n!}{i!(n-i)!}$.

Example 3.9

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \tag{22}$$

Question 3.10. Consider the expression

$$(1+x)(1+x^2)(1+x^3)...(1+x^7)(1+x^8)(1+x^9)(1+x^{10})$$

What is the coefficient of x^9?

Hint: how many ways can you make 9 from the indices?

Summations is another topic that likes to frequent the PAT - though we assume familiarity, we'll talk you through some tricks that will help evaluating them.

The first trick is *partial fraction decomposition*. We won't go into it formally as it is quite complicated, but we'll show you a through examples so you can get an idea of how to use it. Partial fraction decomposition is also very useful when evaluating integrals, and it's something you should always keep in mind to simplify fractions with seemingly complex polynomials.

Theorem 3.11 (Partial Fraction Decomposition - general example)

$$\frac{f(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d} \tag{23}$$

Example 3.12 (Slightly different to our general case)

$$\begin{aligned}
\frac{x^2-3}{(x^2+2)(x-1)} &= \frac{Ax+B}{x^2+2} + \frac{C}{x-1} \\
&\Rightarrow x^2-3 = (Ax+B)(x-1) + C(x^2+2) \\
\text{When } x=1: &\ -2 = 3C \Rightarrow C = -\frac{3}{2} \\
\text{When } x=0: &\ -3 = -B + 2C \Rightarrow B = 0 \\
\text{When } x=-1: &\ -2 = 2A - \frac{9}{2} \Rightarrow A = \frac{5}{4} \\
\therefore \frac{x^2-3}{(x^2+2)(x-1)} &= \frac{5x}{4(x^2+2)} - \frac{3}{2(x-1)}
\end{aligned} \tag{24}$$

Corollary 3.13

This is very useful for integrating:

$$\begin{aligned}\int \frac{x^2-3}{(x^2+2)(x-1)}\mathrm{d}x &= \int \frac{5x}{4(x^2+2)}\mathrm{d}x - \int \frac{3}{2(x-1)}\mathrm{d}x \\ &= \frac{5}{8}\ln(x^2+2) - \frac{3}{2}\ln(|(x-1)|) + C\end{aligned} \tag{25}$$

Example 3.14

What is the value of

$$\sum_{n=1}^{99} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{99\cdot 100} \tag{26}$$

Solution. First we use partial fractions to make the $\frac{1}{n(n+1)}$ more manageable.

$$\begin{aligned}\frac{1}{n(n+1)} &= \frac{A}{n} + \frac{B}{n+1} \\ 1 &= A(n+1) + Bn \\ n = 0 \Rightarrow A &= 1 \\ n = -1 \Rightarrow B &= -1 \\ \therefore \frac{1}{n(n+1)} &= \frac{1}{n} - \frac{1}{n+1}\end{aligned} \tag{27}$$

Hence we have

$$\sum_{n=1}^{99} \frac{1}{n(n+1)} = \sum_{n=1}^{99} \frac{1}{n} - \frac{1}{n+1}$$

We now introduce the idea of telescoping sums. These are summations whose terms cancel out as you evaluate the series. Take a look at this seres for example:

$$\sum_{n=1}^{99} \frac{1}{n} - \frac{1}{n+1} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right)$$

Now we see that our sum simplifies too

$$\begin{aligned}\sum_{n=1}^{99} \frac{1}{n(n+1)} &= 1 - \frac{1}{100} \\ &= \frac{99}{100}\end{aligned} \tag{28}$$

□

Here's some things you should already know about arithmetic and geometric series:

Definition 3.15 (Arithmetic Sequence)**.** We define an *Arithmetic Sequence*, a_n, to be $a_n = a_1 + (n-1)d$, where a_1 is the first term in the sequence, n is the n^{th} term in the sequence, and d is the difference between consecutive terms.

Theorem 3.16 (Arithmetic Series)

$$\sum_{i=1}^{n} a_i = \frac{n}{2}\left(2a_1 + (n-1)d\right)$$
$$= \frac{n(a_1 + a_n)}{2} \tag{29}$$

Remark 3.17
For the final equality, we are saying the sum of an arithmetic series is equal to the mean of the first term in the sequence plus the last term in the sequence.

Definition 3.18. We define a *Geometric Sequence*, b_n, to be $b_n = b_1 r^{n-1}$. Where we call r the common ratio.

Theorem 3.19 (Geometry Series)

$$\sum_{i=1}^{n} b_i = \frac{b_1(r^n - 1)}{r - 1} \tag{30}$$

Note that if $|r| < 1$, then each term in the geometric series gets smaller, and if we were to evaluate this *forever* i.e take the limit to *infinity*, the series would converge to a value

Theorem 3.20 (Infinte Geometric Sequence)

$$\sum_{n=1}^{\infty} b_n = \frac{b_1}{1 - r} \tag{31}$$

Exercise 3.21. Try to derive all these given results for yourself!

There are some other special case you may want to keep in mind:

Theorem 3.22 (sum of n)

$$\sum_{i=1}^{n} i = \frac{n}{2}(n+1) \tag{32}$$

Proof. This is simply just an arithmetic series with a common difference of 1 and $a_1 = 1$. □

Theorem 3.23 (sum of n^2)

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{33}$$

The proof of this isn't particularly well motivated, so it's not something you're expected to know, but it's interesting anyway.

Proof. Consider

$$\begin{aligned}(i-1)^3 &= i^3 - 3i^2 + 3i - 1 \\ i^3 - (i-1)^3 &= 3i^2 - 3i + 1 \\ \Rightarrow \sum_{i=1}^{n}\left[i^3 - (i-1)^3\right] &= \sum_{i=1}^{n}\left[3i^2 - 3i + 1\right]\end{aligned} \tag{34}$$

The left hand-side is simply a telescopic series, hence:

$$\begin{aligned} n^3 &= 3\left(\sum_{i=1}^{n} i^2\right) - 3\left(\sum_{i=1}^{n} i\right) + \left(\sum_{i=1}^{n} 1\right) \\ &= 3\left(\sum_{i=1}^{n} i^2\right) - \frac{3n}{2}(n+1) + n \\ \sum_{i=1}^{n} i^2 &= \frac{n^3}{3} + \frac{n}{2}(n+1) - \frac{n}{3} \\ &= \frac{n(n+1)(2n+1)}{6}\end{aligned} \tag{35}$$

To finish off this section, we will consider some more unusual series that you might want to keep in mind, though it's probably not going to come up:

Theorem 3.24 (Product of Sines)

$$\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)\cdots\sin\left(\frac{(n-2)\pi}{n}\right)\sin\left(\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}} \tag{36}$$

Theorem 3.25 (Product of Cosines)

$$\cos\left(\frac{\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right)\cos\left(\frac{3\pi}{n}\right)\cdots\cos\left(\frac{(n-2)\pi}{n}\right)\cos\left(\frac{(n-1)\pi}{n}\right) = \frac{\sin\left(\frac{n\pi}{2}\right)}{2^{n-1}} \tag{37}$$

Theorem 3.26 (Product of Tangents)

$$\tan\left(\frac{\pi}{n}\right)\tan\left(\frac{2\pi}{n}\right)\tan\left(\frac{3\pi}{n}\right)\cdots\tan\left(\frac{n-2\pi}{n}\right)\tan\left(\frac{n-1\pi}{n}\right) = \frac{n}{\sin\left(\frac{n\pi}{2}\right)} \tag{38}$$

Theorem 3.27 (Lagrange Identities)

$$\sum_{k=1}^{n}\sin(k\theta) = \frac{1}{2}\cot\left(\frac{\theta}{2}\right) - \frac{\cos\left(\left(n+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)} \tag{39}$$

$$\sum_{k=1}^{n}\cos(k\theta) = \left(\frac{\theta}{2}\right) - \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)} - \frac{1}{2} \tag{40}$$

□

§4 Calculus

The calculus required for the PAT isn't usually difficult in and of itself - but this is largely because more difficult questions will require you to build on your knowledge. It's for this reason that you should be very confident with your calculus skills before sitting the exam.

To begin with, one key technique you should know is how to differentiate and integrate polynomials of various powers. Take, for example, this scenario:

Example 4.1

Find $\frac{d}{dx}\left(1+2x^3\right)^{-3/2}$

This is found by applying the chain rule. We know that

$$\frac{d}{dx}f\left(g\left(x\right)\right)=f'\left(g\left(x\right)\right)g'\left(x\right)$$

where in this case, $f\left(u\right)=u^{-\frac{3}{2}}$ and $g\left(u\right)=1+2u^3$. So, we see that:

$$\frac{d}{dx}f\left(g\left(x\right)\right)=-\frac{3}{2}\left(1+2x^3\right)^{-\frac{5}{2}}6x^2=-9x^2\left(1+2x^3\right)^{-\frac{5}{2}}$$

Another thing which you should be comfortable with is integrating these expressions. An important technique is to figure out how to 'add 0' to simplify expressions.

Theorem 4.2

Suppose you have an expression

$$\int\frac{ax+b}{cx+d}\,dx$$

This is not easily integrable, unless we do this neat trick:

$$\begin{aligned}\frac{ax+b}{cx+d}&=\frac{k\left(cx+d\right)+e}{cx+d}\\&=k+\frac{e}{cx+d}\end{aligned}$$

for some k, e which must be found. Then, this expression is easily integrable. This method is applicable to expressions involving polynomials of higher orders as well, but it just means you need to determine more unknown coefficients.

To put this in action, consider the following example:

Example 4.3

$$\int\frac{12x^2+10x+4}{3x^2+2x+1}\,dx$$

This can be rewritten as:

$$\int\frac{2\left(3x^2+2x+1\right)+6x+2}{3x^2+2x+1}\,dx$$

which simplifies to:

$$\begin{aligned}&\int 2+\frac{6x+2}{3x^2+2x+1}\,dx\\&=2x+\ln\left(\left|3x^2+2x+1\right|\right)+c\end{aligned}$$

I'll just be going over some tips and tricks to help reduce the amount of time you spend doing questions, and also to make questions easier.

Example 4.4

Often, for products of trig functions you'll need to use this method to express the integral of a function to the n^{th} power to one of a lower order. To do this, its often best to use reduction formulae to do so. For example, let's find the reduction formula for

$$I_n = \int \cos^n(x)dx$$

We can write this as

$$I_n = \int \cos^{n-1}(x)\cos(x)dx$$

and then apply integration via parts (IBP). So:

$$\begin{aligned} I_n &= \cos^{n-1}(x)\sin(x) + (n-1)\int \sin^2(x)\cos^{n-2}(x)\,dx \\ &= \cos^{n-1}(x)\sin(x) + (n-1)\int (1-\cos^2(x))\cos^{n-2}(x)\,dx \\ &= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx - (n-1)\int \cos^n(x)\,dx \\ &= \cos^{n-1}(x)\sin(x) + (n-1)I_{n-2} - (n-1)I_n \end{aligned}$$

and thus, we see that:

$$I_n = \frac{1}{n}\left(\cos^{n-1}(x)\sin(x) + (n-1)I_{n-2}\right)$$

A similar method can be used for a great many other problems, usually dealing with trig functions as mentioned before.

Theorem 4.5 (Weierstrass Substitution (t-substitution))

Often, for tricky integrals of trigonometric functions, a useful substitution to do is the t-sub. This involves letting

$$t = \tan\left(\frac{x}{2}\right)$$

Thus, we have:

$$\int f(\cos(x),\sin(x))dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}dt$$

Perhaps the most common way of using calculus within your physics course up til now has been to work out rates of change (and often minima and maxima) of various functions.

One nice trick which you can use to speed up calculations is differentiating functions implicitly.

Theorem 4.6

Suppose we have a function $f(x,y)$, and we wish to find $\frac{dy}{dx}$. To do this implicitly, let us define:

- $\frac{\partial f}{\partial x}$ as the *partial derivative with respect to* x of f. For your purposes, all this means is to differentiate f treating y as a constant variable.
- $\frac{\partial f}{\partial y}$ as the *partial derivative with respect to* y of f. This similarly means treat x as a constant.

Then, assume we have:

$$f(x,y) = c$$
$$f(x,y) - c = 0$$

and so differentiating with respect to x, we see that:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

Where the second term was introduced via the chain rule (as y itself, is a function of x).

For a better explanation of where the $\frac{dy}{dx}$ comes from, consider:

$$\frac{d}{dx} f(y(x)) = \frac{df}{dy} y'(x)$$

The reasons we have the *partial* differentiation sign in the equations in implicit differentiation is because f is a function of *both* x and y.

This may all seem a little abstract, so lets try out an example.

Example 4.7

What is the tangent line to the circle with radius 5, centred at the origin, at the location $(3,4)$?

We have the equation of the circle in the form $f(x,y) = c$ as:

$$x^2 + y^2 = 25$$

and so we have to put it in the form with partial derivatives. To find $\frac{\partial f}{\partial x}$, lets treat y as a constant and differentiate with respect to x to get:

$$\frac{\partial f}{\partial x} = 2x$$

similarly, we have

$$\frac{\partial f}{\partial y} = 2y$$

and so we have:

$$2x + 2y\frac{dy}{dx} = 0$$

which we can rearrange to get

$$\frac{dy}{dx} = -\frac{x}{y}$$

and so the equation of the tangent line at the point specified is:

$$y - 4 = -\frac{3}{4}(x-3)$$

You should note, you aren't expected to know about partial derivatives in any detail. All you have to remember to do is to first differentiate the function treating y as a constant, and then treating x as a constant (and adding on a $\frac{dy}{dx}$ after any time you differentiate with respect to y).

Now, lets take a look at turning points.

Theorem 4.8

For a given function of y and x, we know that at any given turning point $y'(x) = 0$, but in order to determine the nature of the turning points we need to follow the following rules:

- minima when $y' = 0$, $y'' > 0$
- maxima when $y' = 0$, $y'' < 0$

Lastly, we have to take a look at symmetry arguments when integrating even and odd functions. But to begin with, we have to define these terms:

Theorem 4.9

A function $f(x)$ is *odd* if:

$$f(-x) = -f(x)$$

and a function is *even* if:

$$f(-x) = f(x)$$

A good example of the former is $\sin(x)$ and of the latter is $\cos(x)$

A direct consequence of this definition is, that:

- For $f(x)$ even:
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$
- And for $f(x)$ odd:
$$\int_{-a}^{a} f(x)\,dx = 0$$

A quick and messy way to check if a function is odd or even, besides inputting x and $-x$ is to see if the taylor series (or indeed the function itself) is comprised of even or odd powers of x.